• Lecture 9: Central forces

$$\vec{F} = f(r)\,\hat{r}$$

- One of the most fascinating topics in classical physics.
- We will apply Newtonian physics to the general problem of central force motion.
- First → general feature of a system of two particles interacting with a central force $f(r)\,\hat{r}$.
- We will see how to obtain a complete solution of the central force problem by employing conservation laws.
- Finally we will apply these concepts to understand planetary motion.

• Central force motion as one body problem:

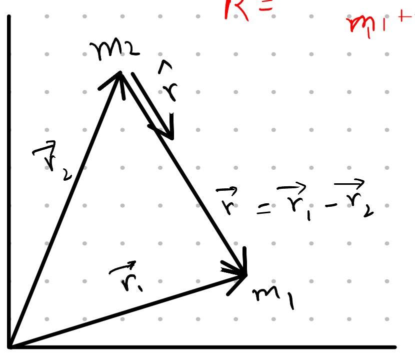
- Two body problem: where two particles are involved with or without interaction.
- Two body problem is difficult to solve, because the system will involve many eq^n s.
- So we will do some awesome trick to convert the two body problem to a one body problem.
- Let us consider, an isolated system consisting of two particles interacting under central force $f(r)$.
- Masses of the particles m_1 and m_2 and position vectors $\vec{r}_1$ and $\vec{r}_2$ respectively.
- So, we have,

$$R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

magnitude, $r = |\vec{r}|$

$$= |\vec{r}_1 - \vec{r}_2|$$



- The equations of motion for both the particles are,

$$m_1 \ddot{\vec{r}}_1 = f(r)\,\hat{r} \quad \text{———} \quad ①$$

$$m_2 \ddot{\vec{r}}_2 = -f(r)\,\hat{r} \quad \text{———} \quad ②$$

$\ddot{r} = \frac{d^2 r}{dt^2} = \vec{a}$

and,
attractive, $f(r) < 0$
repulsive $f(r) > 0$

- Now, solving the two equations will solve the whole system. So it is easy. Isn't it?
- No. Eq^n ① and ② are coupled together by $\vec{r} = \vec{r}_1 - \vec{r}_2$. So both the eq^{n}s needed to solved simultaneously. This makes things difficult.
- But, there is an easy way. Replace $\vec{r}_1$ and $\vec{r}_2$ by $\vec{r}$ and center of mass vector $\vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2)/(m_1+m_2)$. In this case, we will again have two eq^{n}s, but the eq^n of motion for $\vec{R}$ is trivial as there are no external forces.
  - Eq^n for $\vec{r}$ is simple, looks like that of a single particle.

- So now, let's check the eq^n for $\vec{R}$ first.

  No ext. force, so,

$$\ddot{\vec{R}} = 0$$

  Solution is, $\vec{R} = \vec{R}_0 + \vec{v}t$

  - $\vec{R}_0$ and $\vec{v}$ are constants and they depend upon, the choice of coordinate system and initial cond^n.
  - If the origin is at center of mass, then, $\vec{R}_0 = 0$, $\vec{v} = 0$.

- Now, the eq^n for $\vec{r}$. Divide eq^n ① by m_1

$$\ddot{\vec{r}}_1 = \frac{1}{m_1} f(r)\,\hat{r}$$

Divide eqⁿ (2) by m_2,

$$\ddot{\vec{r}}_2 = -\frac{1}{m_2} f(r)\,\hat{r}.$$

Now subtract,

$$\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) f(r)\,\hat{r}$$

$$\Rightarrow \ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = \left(\frac{m_1+m_2}{m_1 m_2}\right) f(r)\,\hat{r}$$

$$\Rightarrow \underbrace{\left(\frac{m_1 m_2}{m_1+m_2}\right)}_{\mu}\underbrace{\left(\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2\right)}_{\ddot{\vec{r}}} = f(r)\,\hat{r}$$

(μ = reduced mass)

$$\Rightarrow \boxed{\mu\ddot{\vec{r}} = f(r)\,\hat{r}} \quad \text{—— (3)}$$

So this eqⁿ is identical to a one particle eqⁿ under force $f(r)$ and mass μ.

- So we have reduced our two body problem to a one body problem.

- The problem now is to find $\vec{r}$ as a function of time from eqⁿ (3). Once we know $\vec{r}$, we can find $\vec{r}_1$ and $\vec{r}_2$ from →

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \quad \text{—— (i)}$$

and

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} \quad \text{—— (ii)}$$

$$\mu\ddot{\vec{r}} = f(r)\,\hat{r}$$

$$\ddot{\vec{R}} = 0$$

Solving for $\vec{r}_1$ and $\vec{r}_2$ gives,

$$\vec{r}_1 = \vec{R} + \left(\frac{m_2}{m_1+m_2}\right)\vec{r}$$

$$f(r) = \frac{k}{r^2}$$

$$\vec{r}_2 = \vec{R} - \left(\frac{m_1}{m_1+m_2}\right)\vec{r}$$

- Therefor, given the force function $f(r)$, we can fully solve the problem of central force.

• General properties of central force motion :

- $\mu \ddot{\vec{r}} = f(r)\hat{r}$ is a vector eq^n. Three eq^{n}s to solve.
- But we can use conservation laws to find some general properties of the sol^n. This will lead us to a eq^n with only one scalar variable.

① The motion is confined to a plane.

$\cancel{\tau}^0 = \frac{dL}{dt} = 0$
$L = C$

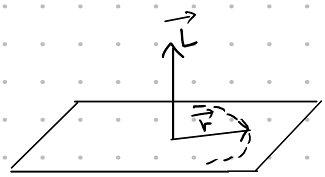
- central force $f(r)\hat{r}$ is in radial direction only. Hence it can exert no torque on μ.
- So the angular momentum $\vec{L}$ of μ is constant.
- This means: the motion is confined to a plane.

$$\vec{L} = \vec{r} \times \vec{p}$$

$L = r p \sin\theta$

$$= \vec{r} \times \mu \vec{v} \qquad , \quad v = \frac{dr}{dt} = \dot{r}$$

So $\vec{r}$ is perpendicular to $\vec{L}$.
And $\vec{L}$ is fixed (constant). So $\vec{r}$ can only move in a plane perpendicular to $\vec{L}$.

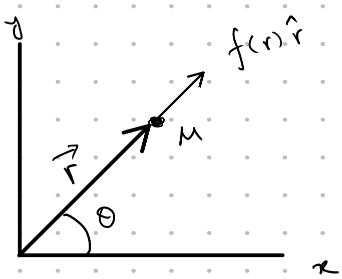


- Since the motion is confined in a plane we can choose our coordinates, so that the motion is in xy pane. In polar coordinates. $\mu \ddot{\vec{r}} = f(r)\hat{r}$ becomes,

$$\mu(\ddot{r} - r\dot{\theta}^2) = f(r) \quad \text{—— ④}$$

(r, θ)

$$\mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \quad \text{—— ⑤}$$



② Energy and angular momentum are constants of motion.

- We have reduced the problem to two dimensions by using the fact the the direction of $\vec{L}$ is constant.
- Other two important constants → magnitude of angular momentum $l \equiv |\vec{L}|$ and Energy E.

- Using these two constants make central force motion very insightful.

Magnitude of angular momentum

$$l = \mu r v_\theta = \mu r^2 \dot{\theta} \quad \Rightarrow \dot{\theta}^2 = \frac{l^2}{\mu^2 r^4}$$

$f(r) \cdot U$

Energy,

$$E = \frac{1}{2}\mu v^2 + U(r)$$

$$= \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U(r)$$

Wher, $U(r) = \int_{r_a}^{r} f(r)\, dr$, $U(r_a) = 0$

is the potential energy.

- Check the energy eq^n,

$\ddot{r} = \frac{f(r)}{\mu}$

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 + U(r)$$

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r)$$

This looks like eq^n of motion of a particle in one dimension. All reference to θ is gone.

Now,

$$E = \frac{1}{2}\mu\dot{r}^2 + U_{eff} \quad \text{--- (6)}$$

$\Omega \times (\Omega \times r)$

- Where $U_{eff} = \frac{1}{2}\frac{l^2}{\mu r^2} + U(r)$ is the effective potential.

- The extra term in potential is the centrifugal potential.

- Now a formal solution of (6).

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l}{\mu r^2} + U(r)$$

$$\Rightarrow \frac{dr}{dt} = \left\{\frac{2}{\mu}(E - U_{eff})\right\}^{1/2} \quad \text{--- (a)}$$

$$\Rightarrow t - t_0 = \int_{t_0}^{t} dt = \int_{r_0}^{r} \frac{dr}{\left\{(2/\mu)(E - U_{eff})\right\}^{1/2}} \quad \text{--- (7)}$$

$t = g(r)$

– Eq^n (7) gives r as a function of t.

– We can also find θ as a function of t.

(b) $\frac{d\theta}{dt}$ / (a) $\frac{dr}{dt}$ $= \frac{d\theta}{dr}$

$$\frac{d\theta}{dt} = \frac{l}{\mu r^2} \quad \text{— (b)}$$

$$\Rightarrow \theta - \theta_0 = \int_{t_0}^{t} \frac{l}{\mu r^2}\, dt$$

(because r is known as a function of t)

– So we have got r(t) and θ(t). We can also obtain r(θ), the orbit of the particle. Dividing (b) by (a),

$$\frac{d\theta}{dr} = \frac{l}{\mu r^2} \times \frac{1}{\left\{(2/\mu)\left(E - U_{eff}\right)\right\}^{1/2}}$$

- This completes the formal solution of central force problem. We can obtain r(t), θ(t) and r(θ) once we get a potential U(r).