• Lecture 8: Physics in a rotating system (includes coriolis and centripetal accⁿ)

- The transformation from an inertial coordinate system to a rotational system is fundamentally different from the transformation to a translating system.

- A coordinate system translating uniformly relative to an inertial system is also inertial. The transformation leaves the laws of motion unaffected.

- A uniformly rotating system is intrinsically non inertial. Because rotational motion is accelerated motion. So laws of motion will always involve fictious forces, such as coriolis force and centripetal force.

- These forces do not have the simple form of uniform gravitational force. Involves several non-trivial terms.

- Inspite of these complications, rotating systems are very helpful.

- Example: Physics of airflow over the surface of the earth. It is easier to explain the rotational motion of weather systems in terms of fictious forces than to use inertial coordinates on the rotating earth.

- Now lets talk about physics in rotating frames:

  - Imagine a particle of mass "m" accelerating at rate $\vec{a}$ with respect to inertial coordinates and at rate $\vec{a}_{rot}$ with respect to a rotating coordinate system, then the eq^n of motion in the inertial system is,

$$\vec{F} = m\vec{a}.$$

In the rotating coordinate system,

$$\vec{F}_{rot} = m\vec{a}_{rot}$$

  - If the acc^n in the two coordinate systems are related by

$$\vec{a} = \vec{a}_{rot} + \vec{A}$$

where $\vec{A}$ is relative acc^n, then,

$$\vec{F}_{rot} = m(\vec{a} - \vec{A}) = \vec{F} + \vec{F}_{fict}$$

where $\vec{F}_{fict} = -m\vec{A}$

So our task is to find $\vec{A}$.

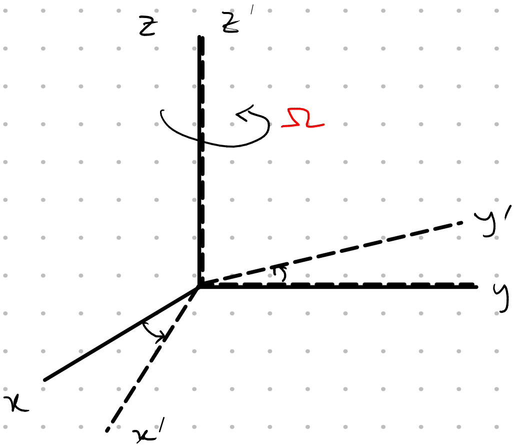
• **Basic idea:** Find the transformation rule relating the time derivative of any vector in inertial and rotating coordinates.

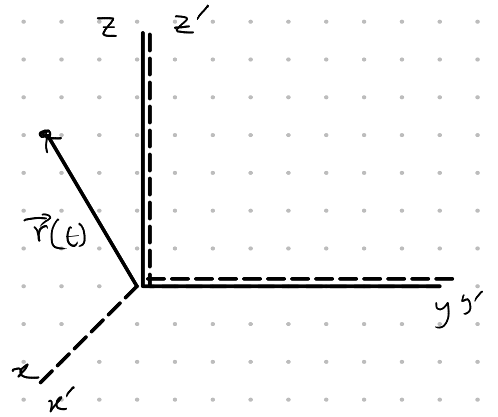
• **Time derivatives and rotating coordinates:**

– We are interested in pure rotation without translation. So imagine a rotating system (x', y', z') whose origin coincides with the origin of an inertial system (x, y, z).

(x, y, z) (x', y', z)

m, $\underline{\vec{r}(t)}$, $\vec{r}'(t)$



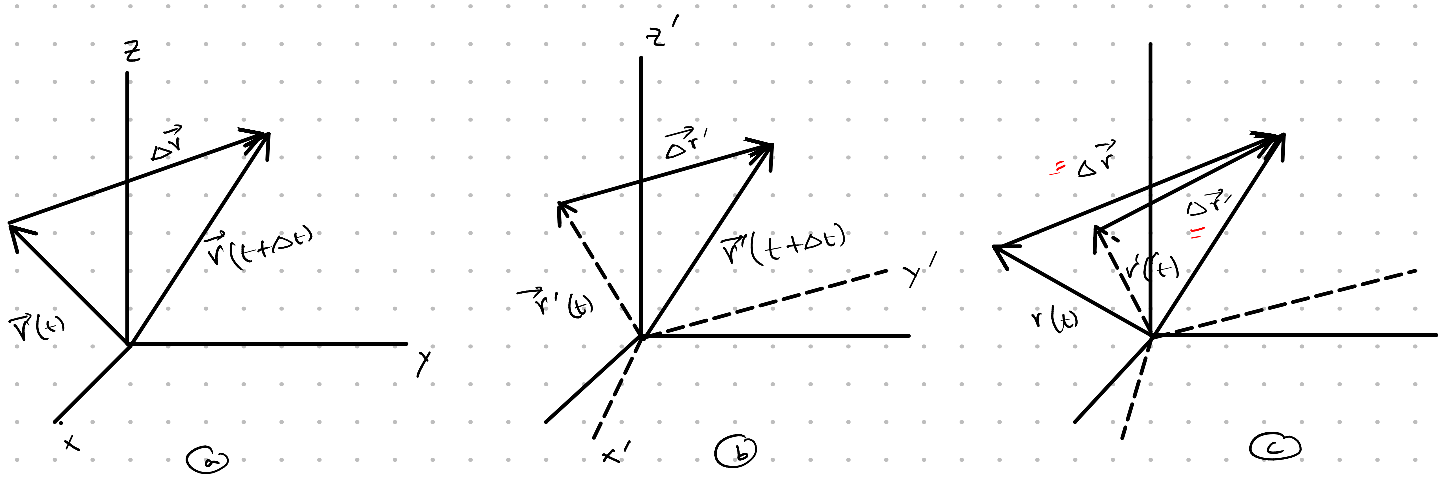


• Axis of rotation z' coincides with z axis. So angular velocity Ω of the rotating system lies in the z direction.

• Now let x axis and x' axis coincide instantaneously at time t.

• Also suppose a particle has position vector $\vec{r}(t)$ in the xz plane ($x'z'$ plane) at time t.

• At time $t + \Delta t$, the position vector is $\vec{r}(t+\Delta t)$, and displacement of the particle is $\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t)$. (figure ⓐ)



- For an observer at the rotating frame, the final position vector is also $\vec{r}(t+\Delta t)$. But initial position vector in $x'z'$ plane is $r'(t)$. The displacement measured is,

$$\Delta r' = r(t+\Delta t) - r'(t) \qquad \text{(figure (b))}$$

- Now $x'z'$ plane is rotated away from its earlier position. So from figure (c), $\Delta\vec{r}$ and $\Delta\vec{r}'$ is not the same.

$$\Delta\vec{r} = \Delta\vec{r}' + \vec{r}'(t) - \vec{r}(t).$$

Consequently the velocity is different in these two frames.

- $\vec{r}(t)$ and $\vec{r}'(t)$ only differ by a pure rotation and we know from previous lecture that, $\frac{d\vec{r}}{dt} = \vec{\omega}\times\vec{r}$. We use this.

Such that, $\vec{r}'(t) - \vec{r}(t) = (\vec{\Omega}\times\vec{r})\,\Delta t$ $\qquad$ $\underline{\underline{V = \omega \times r}}$

Hence,

$$\Delta\vec{r} = \Delta\vec{r}' + \vec{r}'(t) - \vec{r}(t)$$

$$\Rightarrow \Delta\vec{r} = \Delta\vec{r}' + (\vec{\Omega}\times\vec{r})\,\Delta t$$

$$\Rightarrow \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta\vec{r}'}{\Delta t} + (\vec{\Omega}\times\vec{r})$$

Now taking the limit $\Delta t \to 0$

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}'}{dt} + (\vec{\Omega}\times\vec{r})$$

$$\Rightarrow \vec{V}_{in} = \vec{V}_{rot} + (\vec{\Omega}\times\vec{r})$$

- An alternative way is $\left(\frac{d\vec{r}}{dt}\right)_{in} = \left(\frac{d\vec{r}}{dt}\right)_{rot} + (\vec{\omega}\times\vec{r})$ $\qquad$ — (1)

- Since our proof only uses geometric property of $\vec{r}$, eq^n (1) is valid for any vector.

$\vec{B} \Rightarrow$ any vector

$$\left(\frac{d\vec{B}}{dt}\right)_{in} = \left(\frac{d\vec{B}}{dt}\right)_{rot} + (\vec{\Omega}\times\vec{r}) \qquad \text{— (2)}$$

- (Keep in mind: $\vec{B}$ is instantaneously same in both the coordinate systems, only the time rate of change differ).

Acceleration relative to rotating coordinates:

- We can use eq^n (2) to relate accl^n observed in a rotating system $\vec{a}_{rot} = \left(d\vec{v}_{rot}/dt\right)_{rot}$ to the accl^n in an inertial system $\vec{a}_{in} = \left(d\vec{v}_{in}/dt\right)_{in}$.

- Now apply eq^n (2) to the velocity vector, $\vec{v}_{in}$.

$$\vec{a}_{in} = \left(\frac{d\vec{v}_{in}}{dt}\right)_{in} = \left(\frac{d v_{in}}{dt}\right)_{rot} + (\vec{\Omega} \times \vec{v}_{in})$$

We know,

$$\vec{v}_{in} = \vec{v}_{rot} + \vec{\Omega} \times \vec{r}$$

So, we have,

$$\vec{a}_{in} = \left[\frac{d}{dt}(\vec{v}_{rot} + \vec{\Omega} \times \vec{r})\right]_{rot} + (\vec{\Omega} \times \vec{v}_{rot}) + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

$$\Rightarrow \vec{a}_{in} = \left(\frac{d}{dt}\vec{v}_{rot}\right) + \vec{\Omega} \times \left(\frac{d\vec{r}}{dt}\right)_{rot} + (\vec{\Omega} \times \vec{v}_{rot}) + (\vec{\Omega} \times (\vec{\Omega} \times \vec{r}))$$

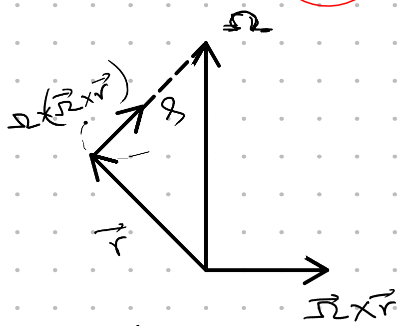
$$\Rightarrow \vec{a}_{in} = \vec{a}_{rot} + 2\vec{\Omega} \times \vec{v}_{rot} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

- Now, let us examine various terms involved in the previous eq^n.

(i) the term $\vec{a}_{rot}$ is just the accl^n observed in the rotating coordinate system.

<u>eg</u>: If we measure the accl^n of a car or a plane in a coordinate system fixed to the rotating earth, we are measuring $\vec{a}_{rot}$.

(ii) the term $\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$:



Note: $\vec{\Omega} \times \vec{r}$ is perpendicular to $\vec{\Omega}$ and $\vec{r}$, and magnitude $\Omega\rho$.

$\rho \rightarrow$ Perpendicular distance from the axis to the tip of $\vec{r}$.

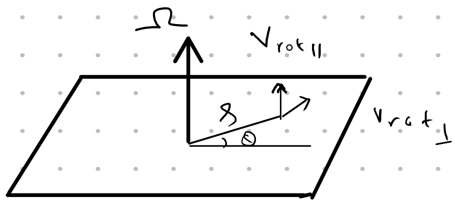
Hence $\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$ is directed radially inwards.

Magnitude of $\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$ is $\Omega^2 \rho$.

→ It is centripetal acceleration, arising because every point at rest in the rotating system is actually moving in a circular path in inertial space.

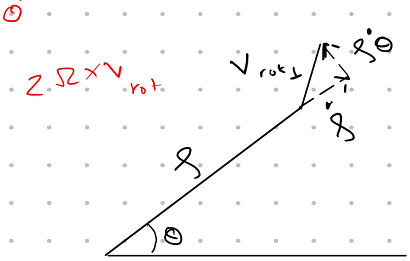
(iii) The term $2\vec{\Omega}\times\vec{v}_{rot}$: It is general vector expression for the coriolis acceleration in 3D.

- Resolve $\vec{v}_{rot}$ into components $\vec{v}_{rot\perp}$ and $\vec{v}_{rot\parallel}$ (to $\vec{\Omega}$)
- Only $\vec{v}_{rot\perp}$ contributes to $2\vec{\Omega}\times\vec{v}_{rot}$
- Hence coriolis accel^n is perpendicular to $\vec{\Omega}$.



Q. How does it arise?

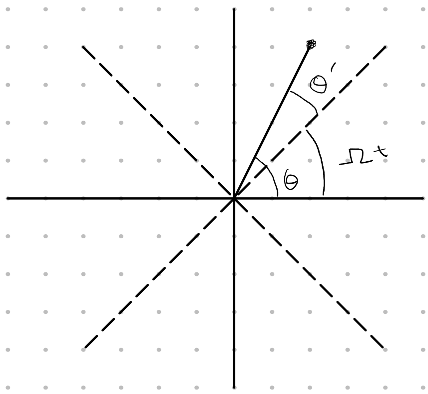
$v = \dot{\rho}\,\hat{r} + \rho\dot{\theta}\,\hat{\theta}$



— The radial component $\dot{\rho}$ of $\vec{v}_{rot\perp}$ contributes $2\Omega\dot{\rho}$ in the tangential direction to $\vec{a}_{in}$.

— The tangential component $\rho\dot{\theta}'$ of $\vec{v}_{rot\perp}$ contributes $2\Omega\rho\dot{\theta}'$ towards the rotation axis.

— in inertial space the angular velocity is $\dot{\theta} = \dot{\theta}' + \Omega$ and centripetal accel^n term in $\vec{a}_{in}$ is,



$a_{in} = (\,-\,-\,)$

$$\rho\dot{\theta}^2 = \rho(\dot{\theta}'+\Omega)^2$$

$$= \rho\dot{\theta}'^2 + 2\Omega\rho\dot{\theta}' + \rho\Omega^2$$

$$\left[\begin{array}{l} \theta = \theta' + \Omega t \\ \dot{\theta} = \dot{\theta}' + \Omega \end{array}\right]$$

the three term on the right → three term on the right of $\vec{a}_{in}$.

$\rho\dot{\theta}'^2 \rightarrow \vec{a}_{rot}$

$2\Omega\rho\dot{\theta}' \rightarrow 2\vec{\Omega}\times\vec{v}_{rot}$

$\rho\Omega^2 \rightarrow \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$

- The apparent force in rotating coordinate system:

We have,

$$\vec{a}_{in} = \vec{a}_{rot} + 2\vec{\Omega} \times \vec{v}_{rot} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

$$\Rightarrow \vec{a}_{rot} = \vec{a}_{in} - 2\vec{\Omega} \times \vec{v}_{rot} - \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

So the force,

$$\Rightarrow \vec{F}_{rot} = m\vec{a}_{rot} = m\vec{a}_{in} - m\left[2\vec{\Omega} \times \vec{v}_{rot} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})\right]$$

$$\Rightarrow \vec{F}_{rot} = \vec{F} + \vec{F}_{fict}$$

Where,

$$\vec{F}_{fict} = \underbrace{\underbrace{-2m\vec{\Omega} \times \vec{v}_{rot}}_{\text{Coriolis force}} \underbrace{- m\vec{\Omega} \times (\vec{\Omega} \times \vec{r})}_{\text{centrifugal force}}}_{\text{non-physical force.}}$$

− +

(arise due to kinematics, not due to physical interactions)

- Centrifugal force increases with ρ. But a real force decreases with increasing distance.
- Seems quite real to an observer in rotating frame.
- eg - Drive a car fast around a curve.

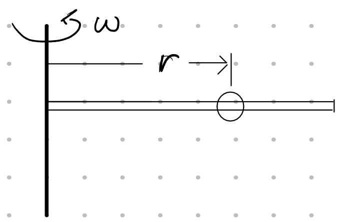
- Example: Coriolis force

A bead slides without friction on a rigid wire rotating at constant angular speed ω. Find the force exerted by the wire on the bead.

→ In a coordinate system rotating with the wire, the motion is purely radial.

→ We will neglect gravity.

- Check the force diagram. (frictionless wire)



- In the rotating coordinate system,

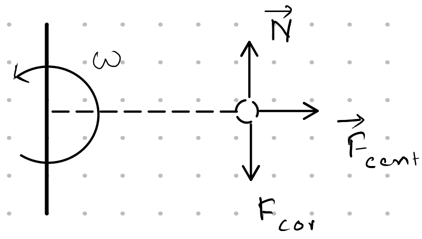
$$F_{cent} = m\ddot{r} \quad \text{— ①}$$

$$N - F_{cor} = 0 \quad \text{— ②}$$

$a = \ddot{r} = \frac{d^2r}{dt^2}$

$m\frac{d^2r}{dt^2} - m\omega^2 r = 0$

We know, $F_{cent} = m\omega^2 r$



① $\Rightarrow m\ddot{r} - m\omega^2 r = 0$

↳ Differential eqⁿ. solⁿ is

$$r = \underbrace{A e^{\omega t} + B e^{-\omega t}}$$

constants (depends on initial condⁿs.)

- The tangential eqⁿ of motion ⇒ no tangential acclⁿ in rotating system,

$2\vec{\Omega} \times \vec{v}$

$\dot{r} = \frac{dr}{dt} = v$

② $\Rightarrow N = F_{cor} = 2m\dot{r}\omega$

$$= 2m\omega^2 (A e^{\omega t} - B e^{-\omega t})$$

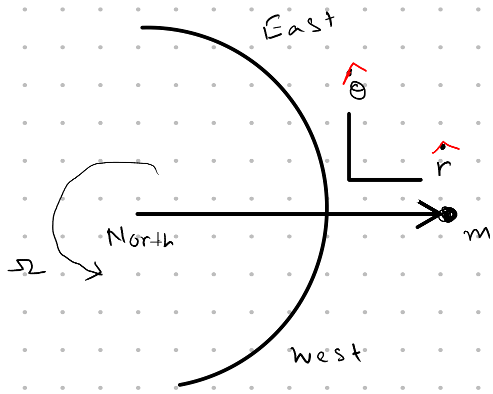
Now given the initial condⁿs, this problem can be fully solved.

• **Example:** Deflection of a falling mass

- Because of the coriolis force, falling objects on earth deflects horizontally.
- We shall calculate the deflection for a mass m dropped from a tower of height h at the equator of earth.

- Consider a coordinate system fixed to earth

So force,



$$\vec{F} = -mg\hat{r} - 2m\vec{\Omega} \times \vec{V}_{rot} - m\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

$$F_\theta = -2m\dot{r}\Omega$$

- Gravitational and centrifugal forces are radial.
- m is dropped from rest, so coriolis force is in the equatorial plane.
- We have, $\vec{V}_{rot} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

Now, $\vec{\Omega} \times \vec{V}_{rot} = \Omega \dot{r} \hat{\theta} - r\Omega\dot{\theta}\hat{r}$

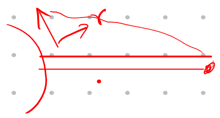
and $\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = -\Omega^2 r \hat{r}$.

We obtain,

$$F_r = -mg + 2m\Omega\dot{\theta}r + m\Omega^2 r$$

$$F_\theta = -2m\dot{r}\Omega$$

F = polar coordinates



Now, the radial eq^n of motion is,

$$m\ddot{r} - mr\dot{\theta}^2 = -mg + 2m\Omega\dot{\theta}r + m\Omega^2 r$$

- Since m falls vertically, so assume $\dot{\theta} \ll \Omega$ Approximation

So we can omit $mr\dot{\theta}^2$ and $2m\Omega\dot{\theta}r$

Thus, $\ddot{r} = -g + \Omega^2 r$ ||

The tangential eq^n of motion is,

$$mr\ddot{\theta} + \underbrace{2m\dot{r}\dot{\theta}}_{\text{small,}} = -2m\dot{r}\Omega$$

$R_e \gg r$

So, $r\ddot{\theta} = -2\dot{r}\Omega$

- Another approximation, during the fall, r changes slightly, from $R_e + h$ to R_e. g is also constant.

So, $r \approx R_e$

So, Radial eq^n, $\ddot{r} = -g + \Omega^2 R_e$

$$= -g'$$

g' = effective gravity accel^n

where $g' = g - \Omega^2 R_e$ is accl^n due to gravity minus centrifugal.

= apparent accl^n dure to gravity.

So, $\ddot{r} = -g' = -g \quad \Rightarrow \dot{r} = -gt$

Solving $\Rightarrow r = r_0 - \frac{1}{2} g t^2$

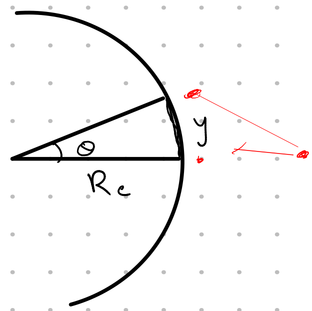
Now, the tangential eq^n,

$$r\ddot{\theta} = -2\dot{r}\Omega$$

$$(r \simeq R_e)$$

$$\Rightarrow r\ddot{\theta} = 2gt\Omega$$

$$\Rightarrow \ddot{\theta} = \frac{2gt\Omega}{R_e}$$



Integrate now,

$$\dot{\theta} = \frac{g\Omega}{R_e} t^2$$

again,

$$\theta = \frac{1}{3} \frac{g\Omega}{R_e} t^3$$

The horizontal deflection is, $y \simeq R_e \theta$, $\sin\theta \simeq \theta$, for $\theta \ll 1$

So, $y = \frac{1}{3} g\Omega t^3$

And time T to fall a distance h,

$$r - r_0 = -h = -\frac{1}{2} g T^2$$

$$\text{So, } T = \sqrt{\frac{2h}{g}}$$

Hence

$$y = \frac{1}{3} g\Omega \left(\frac{2h}{g}\right)^{3/2}$$

– For a tower of height 50 m, $y = 0.77$ cm.