• LECTURE 6 : Gyroscope / Angular momentum conservation

- In the last chapter, we analyzed motion of rigid bodies under fixed axis rotation.
- Now a more general case, where rotation is about any axis.

• vector nature of angular velocity and angular momentum :

- To study rotational system, we need to define a coordinate system.
- What did we do in linear motion?
  - → define a coordinate system
  - → define position of the body $\vec{r}$.
  - → take consecutive derivative of $\vec{r}$ to find $\vec{v}, \vec{a}$.
- Can we do the same in rotational motion?
  - → can we define angular coordinates $\theta_x, \theta_y, \theta_z$ such that,

    angular orientation,

    $$\theta \stackrel{?}{=} \theta_x \hat{i} + \theta_y \hat{j} + \theta_z \hat{k}$$

  - → It doesn't work. Because these angular coordinates are not commutative.

    $$\theta_x \hat{i} + \theta_y \hat{j} \neq \theta_y \hat{j} + \theta_x \hat{i} .$$

eg :

Imagine a pepsi can → rotate it by an angle $\pi/2$ around x-axis. and then $\pi/2$ around y axis.

→ Do the same thing again, but in reverse order.

→ The pepsi can will end up in different orientation after both rotation types.

Hence, $\theta_x \hat{i} + \theta_y \hat{j} \neq \theta_y \hat{j} + \theta_x \hat{i}$. //

However, we can use angular velocity as a perfectly good vector.

$$\vec{\omega} = \frac{d\theta_x}{dt}\hat{i} + \frac{d\theta_y}{dt}\hat{j} + \frac{d\theta_z}{dt}\hat{k}$$

$$= \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

- Reason: Although rotations through finite angles do not commute, infinitesimal rotations do commute.

  So that,

  $\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$ represents a vector.

- Relationship between linear and angular velocity →

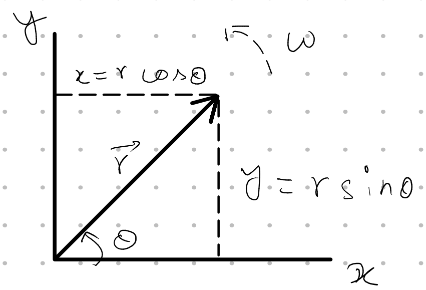
$$\boxed{\vec{v} = \vec{\omega} \times \vec{r}}$$

where $\vec{v} = \frac{d\vec{r}}{dt}$ and $\vec{\omega} = \frac{d\theta}{dt}\cdot\hat{n}$

- Example: Rotation in xy plane

$\vec{\omega} = \omega\hat{k}$

$\vec{r} = x\hat{i} + y\hat{j}$



So $\vec{v} = \vec{\omega} \times \vec{r}$

$= \omega\hat{k} \times (x\hat{i} + y\hat{j})$

$= \omega(x\hat{j} - y\hat{i})$

On plane polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$

So, $\vec{v} = \omega r(\underbrace{\hat{j}\cos\theta - \hat{i}\sin\theta}) $

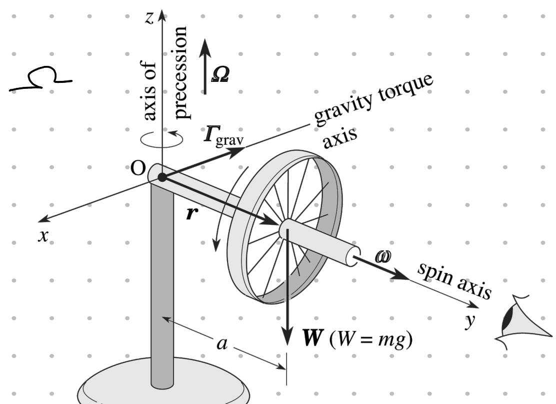
$$\left(\hat{j}\cos\theta - \hat{i}\sin\theta\right) \qquad \hat{\theta}$$

↳ is a unit vector in the tangential θ direction

So, $\vec{v} = \omega r \hat{\theta}$ //

- **The Gyroscope:**

The essentials of a gyroscope are a spinning and a suspension which allows the axle to assume any orientation.



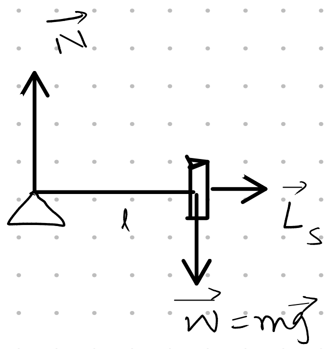
(a)

- If the gyro is released horizontally with one end supported by the pivot, it wobbles off horizontally and then settles down to uniform precession. The axle slowly rotates about the vertical with constant angular velocity Ω. Video?

- Why doesn't the gyro fall?

total vertical force $= \vec{N} - \vec{W}$

If $N = W$, center of mass cannot fall.



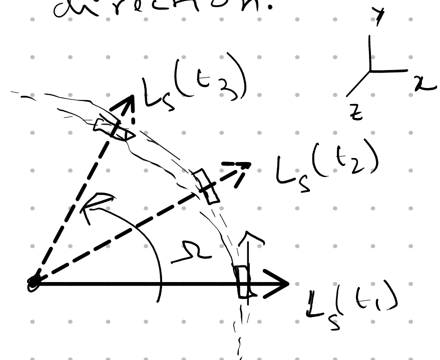
- Remember: Gyro will precess only if it is spinning.

- Angular momentum of Gyro:

$\vec{L}_s \Rightarrow$ Spin angular momentum.
Directed along the axle.
Magnitude $|L_s| = I_0 \omega_s$

$I_0 \rightarrow$ MOI

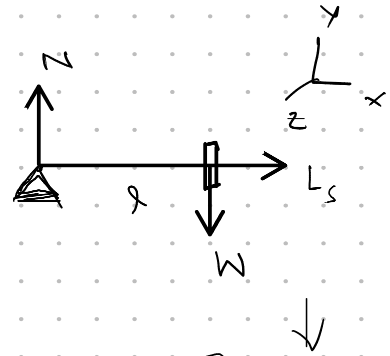
When there is precession, there will be orbital angular momentum, in the z direction. and L_s will rotate,



So, $\left|\frac{dL_s}{dt}\right| = L_s \Omega$

- There must be a torque on the gyro to account for the change in L_s.

- The source of the torque is apparent from the diagram



$\boxed{N - W = 0}$

- pivot is the origin.
- Torque due to W,

$\tau = l W \quad | \quad \tau \rightarrow$ in y direction

We also know,

$$\left|\frac{dL_s}{dt}\right| = \tau = l W$$

and,

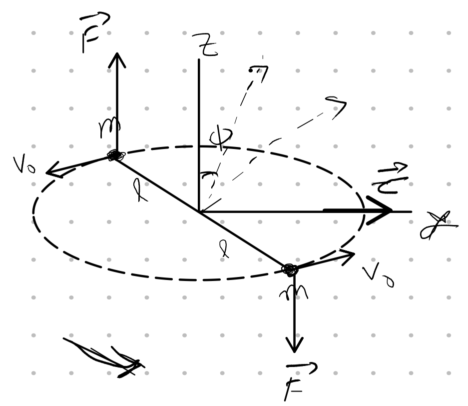
$$\left|\frac{dL_s}{dt}\right| = \Omega L_s = l W$$

$$\therefore \quad \Omega = \frac{l W}{I_0 \omega_s}$$

$$\Omega = \frac{l m g}{I_0 \omega_s}$$

• **Example: Why a gyro precesses (Analogy)**

- Two masses 'm' and 'm'
  - a rod of length $2l$.
  rotating with ang. mom. $\vec{L}_s$ along z.



  - Speed of each mass v_o.

- Consider, torque is applied only of a short interval Δt while the rod is oriented along x axis. The momentum change for each mass,

$$\Delta \vec{p} = m \Delta \vec{v} = \vec{F} \Delta t$$

- $\Delta \vec{v}$ is perpendicular to $\vec{v}_o$, the velocity of each mass changes its direction. and the rod rotates about a new direction.

  - axis of rotation tilt.

$$\Delta \phi \approx \frac{\Delta v}{v_o} = \frac{F \Delta t}{m v_o}$$

- Torque, $\tau = 2Fl$

  Ang. mom. $L_s = 2 m v_o l$

Hence, $$\Delta \phi = \frac{F \Delta t}{m v_o} = \frac{2l \, F \Delta t}{2l \, m v_o}$$

$$= \frac{\tau \Delta t}{L_s}$$

- Rate of precession, $$\Omega = \frac{\Delta \phi}{\Delta t} = \frac{\tau}{L_s}$$ //