LECTURE 5: ANGULAR MOMENTUM (Rigid body motion)

- Till now, we studied the motion of translating bodies - Linear motion.
- But what about bodies in rotational motion? eg: Yoyo/Beyblade?
- In principle we already know that all the particles in the Beyblade follows Newton's laws. So the objective is to just solve those eqns and predict the motion of the Beyblade.
- But that will be incredibly complicated.
- That is why, in this part we'll develope the formalism of rotational motion.

Linear motion	Force	Momentum	Mass
	↓	↓	↓
Rotational motion	Torque	Angular Momentum	Moment of Inertia

(Note: our aim is, of course more ambitious than understanding Beyblades)

Aim: to find a way of analyzing the general motion of a rigid body under any combination of applied forces.

↳ The problem is divided into two simpler problems.

- ↳ Finding center of mass motion
- ↳ Finding the rotational motion about the center of mass.

So we first need to understand: Angular momentum, torque, moment of inertia.

- **Angular Momentum**

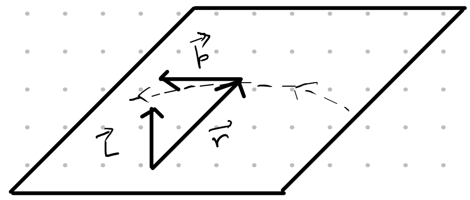
Angular momentum $\vec{L}$ of a particle which has momentum $\vec{p}$ and position vector $\vec{r}$ with respect to a given co-ordinate system is

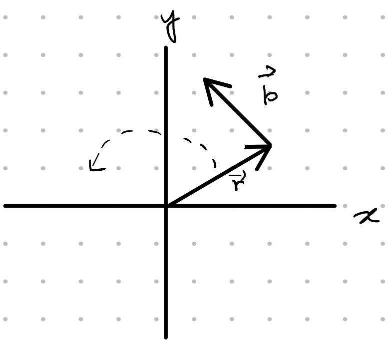
$$\vec{L} = \vec{r} \times \vec{p}$$

unit: $kg.m^2/s$

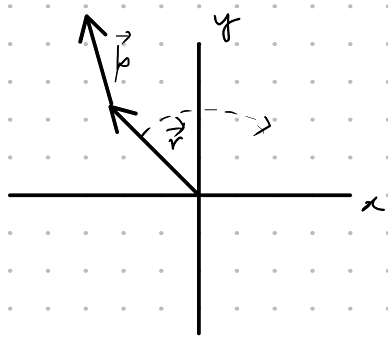
- Angular momentum involves cross product. What about the direction?
- Direction of $\vec{L}$ is perpendicular to both $\vec{r}$ and $\vec{p}$. The plane containing $\vec{r}$ and $\vec{p}$ is called the plane of motion. Direction comes from right hand rule of vector multiplication.

- If $\vec{p}$ and $\vec{r}$ lie on XY plane, $\vec{L}$ will be in the z direction.





$L_z > 0$



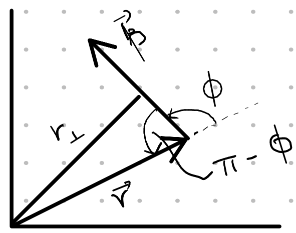
$L_z < 0$

- Method of calculating angular momentum -

$$\vec{L} = \vec{r} \times \vec{p}$$

$$= r p \sin\phi \,\hat{k}$$

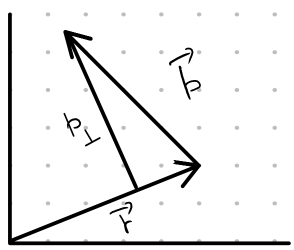
Or $L_z = rp\sin\phi$



We know $r_\perp = r\sin(\pi - \phi) = r\sin\phi$

So, $L_z = r_\perp p$

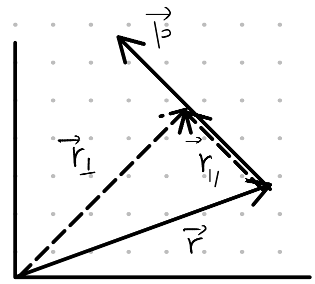
Alternatively, $L_z = r\, p_\perp$



- Another way is by resolving $\vec{r}$ vector.

$\vec{r} = \vec{r}_\perp + \vec{r}_{\parallel}$ → parallel to $\vec{p}$

↳ perpendicular to $\vec{p}$



$\therefore \quad \vec{L} = \vec{r} \times \vec{p} = (\vec{r}_\perp + \vec{r}_{\parallel}) \times \vec{p}$

$= (\vec{r}_\perp \times \vec{p}) + \underbrace{(\vec{r}_{\parallel} \times \vec{p})}_{}$ → zero because parallel.

$= \vec{r}_\perp \times \vec{p}$

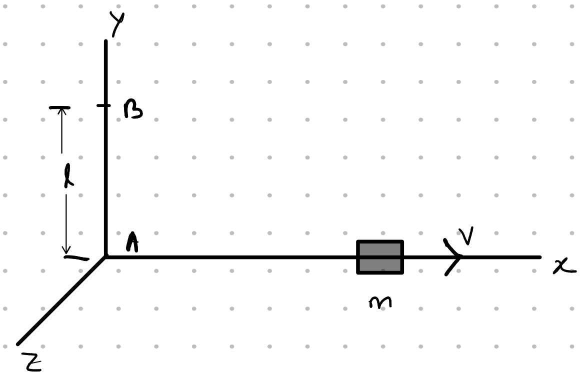
Example: Angular momentum of sliding block.

mass $= m$

velocity $\vec{V} = v\hat{i}$

Angular momentum L_A?

L_B?



Radius vector from A,

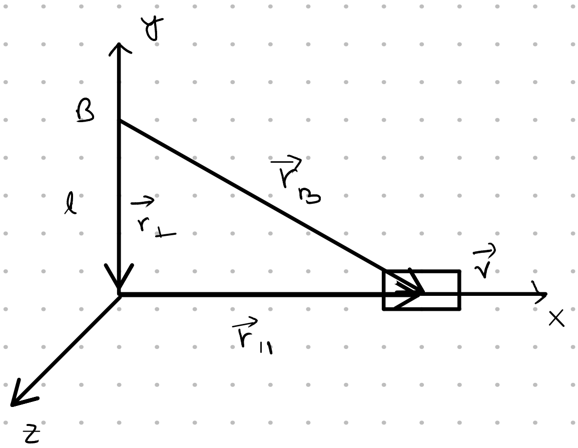
$\vec{r}_A = x\hat{i}$

$\vec{r}_A$ is parallel to $\vec{v}$. So, $L_A = m\vec{r}_A \times \vec{v} = 0$.

Now L_B?

$\vec{r}_{\parallel} \times \vec{v} = 0$

We know, $\vec{r}_\perp \times \vec{v} = lv$



So, $\vec{L}_B = m\vec{r}_B \times \vec{v}$

$= ml v\hat{k}$. //

Or another method

$$L_B = m\vec{r}_B \times \vec{v}$$

$$= m\begin{vmatrix} i & j & k \\ x & -l & 0 \\ v & 0 & 0 \end{vmatrix} = mlv\hat{k}$$

- Torque

Torque due to the force $\vec{F}$ which acts on a particle at position $\vec{r}$, is defined by,

$$\vec{\tau} = \vec{r} \times \vec{F}$$

- similar methods to those of finding angular momentum can also be applied to Torque.

$$|\vec{\tau}| = |\vec{r}_\perp||\vec{F}|$$

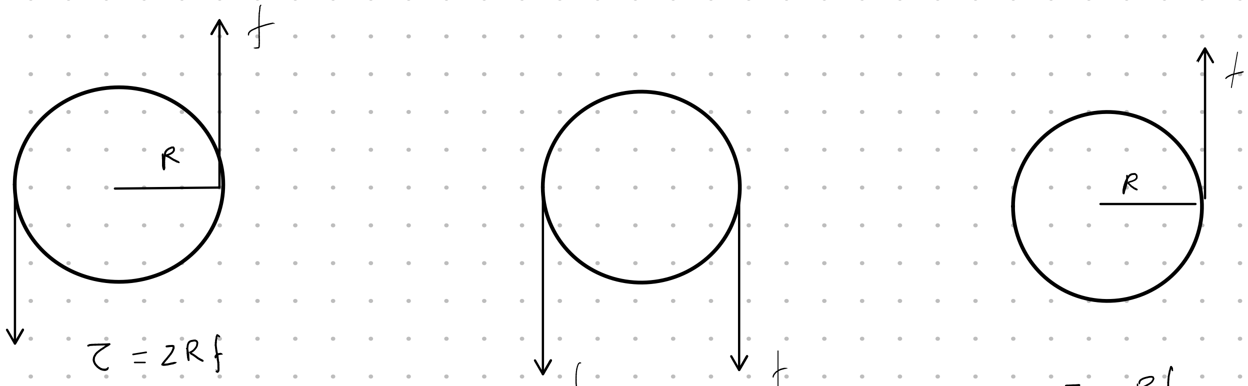
or,

$$|\vec{\tau}| = |\vec{r}||F_\perp|$$

More formally,

$$\vec{\tau} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

- Difference between Torque and Force:
  - Torque depends on the choice of origin but force does not.
  - Direction of force and torque are mutually perpendicular.
  - There can be torque on a system with zero force.



$\tau = 2Rf$ $\qquad$ $\tau = 0$ $\qquad$ $\tau = Rf$

$F = 0$ $\qquad$ $F = 2f$ $\qquad$ $F = f$

- Torque is related to the rate of change of angular momentum.

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$
$$= \left(\frac{d\vec{r}}{dt} \times \vec{p}\right) + \left(\vec{r} \times \frac{d\vec{p}}{dt}\right)$$
$$= (\vec{v} \times \vec{p}) + \left(\vec{r} \times \frac{d\vec{p}}{dt}\right)$$
$$= 0 + (\vec{r} \times \vec{F})$$

$$\frac{d\vec{L}}{dt} = \vec{\tau}$$

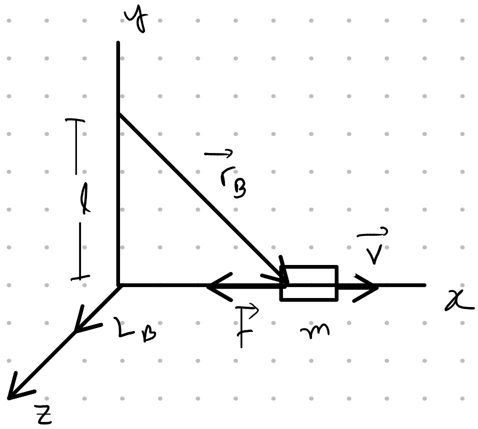
– When $\tau = 0$, $L =$ constant. (conservation of ang. momentum)

- **Example:** Torque on a sliding block.

mass $= m$
$\vec{v} = v\hat{i}$

So, $L_B = m\vec{r}_B \times v$
$= m l v \hat{k}$

- If the block is sliding freely, $\vec{v}$ doesn't change. $\rightarrow L_B$ constant.



Now, suppose there is friction → $\vec{v}$ changes.

$$\vec{f} = -f\,\hat{i}$$

Torque, $\vec{\tau} = \vec{r}_B \times \vec{f} = -l f \hat{k}$

- The block slows down → L_B still along +ve z direction. But magnitude decreases,

So, $\Delta \vec{L}_B$ → −ve z direction.

$$\Delta \vec{L}_B = m l \,\Delta v\, \hat{k} \quad , \quad \Delta v < 0$$

dividing by Δt and taking limit, $\Delta t \to 0$,

$$\frac{d\vec{L}_B}{dt} = m l \frac{dv}{dt}\,\hat{k}$$

$$= -l f\,\hat{k} \qquad , \quad m\frac{dv}{dt} = -f$$

$$\frac{d\vec{L}_B}{dt} = \vec{\tau}_B$$

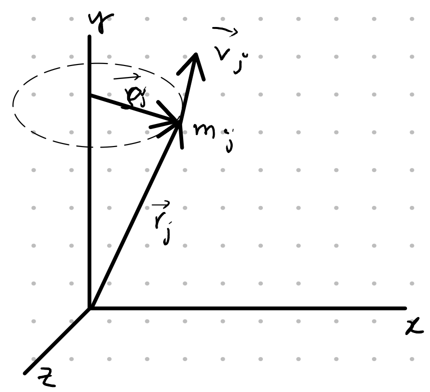
Moment of inertia; fixed axis rotation

- We assume the axis of rotation to be fixed. → z axis.
- So when the rigid body rotates, all the particles of the body remains at a fixed distance from the axis.
- To do so, choose a coodinate system with its origin lying on the axis. $|\vec{r}|$ = constant.
- $\vec{r}$ changes, but $|\vec{r}|$ remains constant.

How? $\vec{v}$ is perpendicular to $\vec{r}$.

$$|\vec{v}_j| = |\vec{r}_j|$$

$$= \omega \rho_j$$



$\rho_j \rightarrow$ Perpendicular distance from the axis to the particle.

$$\rho_j = (x_j^2 + y_j^2)^{1/2}$$

$\omega \rightarrow$ angular velocity.

So, ang. momentum on the j^{th} particle,

$$\vec{L}(j) = \vec{r}_j \times m_j \vec{v}_j$$

component of ang. momentum along the axis

$$L_z(j) = m_j v_j \times (\text{distance to } z \text{ axis})$$

$$= m_j v_j \rho_j$$

We know, $v_j = \omega \rho_j$

So, $L_z(j) = m_j \rho_j^2 \omega$

z-component of total ang. momentum now, is,

$$L_z = \sum_j L_j(z)$$

$$= \sum_j m_j \rho_j^2 \omega$$

Sum is over all the particles of the body.

Previous eqn is also written as,

$$L_z = I \omega$$

Where, $I = \sum_j m_j \rho_j^2$

$I \rightarrow$ moment of Intertia.

$\rightarrow$ depends on both mass distribution in the body and location of the axis.

— For continuous distribution of matter,

$$\sum_j m_j \rho_j^2 \longrightarrow \int \rho^2 \, dm$$

So, $I = \int \rho^2 \, dm = \int (x^2+y^2) \, dm$

to evaluate the integral, replace the mass element, by a product of the density s at the position of dm and the vol^m dV occupied by dm.

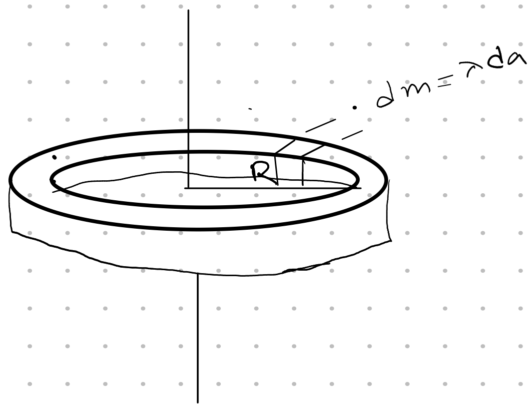
$$dm = s \, dV$$

So, we can write,

$$I = \int \rho^2 \, dm$$

$$= \int (x^2+y^2) \, s \, dV.$$

- Example: MOI of uniform hoop.



Mass = M

Radius = R

$\lambda = M/2\pi R$

$\rho = R$

So,
$$I = \int_0^{2\pi R} R^2 \lambda \, da$$

$$= R^2 \left(\frac{M}{2\pi R}\right) a \Big|_0^{2\pi R}$$

$$I = MR^2$$