LECTURE-3:

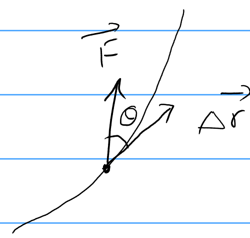
- Integrating eqn of motion in several dimensions:

Force $\vec{F}(\vec{r})$, velocity $\vec{v}$ $|v| = \sqrt{v_x^2+v_y^2+v_z^2}$

- in one dimension, we integrate in simple manner.
- let's generalize this.

- A particle moves a short distance $\Delta\vec{r}$.
- $\Delta\vec{r}$ is so small that, $\vec{F}$ is constant over this displacement.
- taking scalar pdt.

$$\vec{F}\cdot\Delta\vec{r} = m\frac{d\vec{v}}{dt}\cdot\Delta\vec{r} \quad \text{— } (1)$$



Fig

From the diagram,

$$\vec{F}\cdot\Delta\vec{r} = F\,\Delta r\cos\theta$$

- Now, assume: $\vec{v}$ and $\Delta\vec{r}$ are not independent for sufficiently short length of path → $\vec{v}$ is approximately constant.

$$\therefore\ \Delta\vec{r} = \vec{v}\,\Delta t$$

- So we have,

$$m\frac{d\vec{v}}{dt}\cdot\Delta\vec{r} = m\frac{d\vec{v}}{dt}\cdot\vec{v}\,\Delta t$$

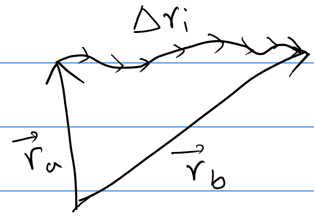
Now using the vector identity, $\vec{A}\cdot\left(\frac{d\vec{A}}{dt}\right) = \frac{1}{2}\left(d/dt\right)\left(A^2\right)$

Prove

$$\vec{v}\cdot\frac{d\vec{v}}{dt} = \frac{1}{2}\frac{d}{dt}\left(v^2\right)$$

Δr_1 , Δr_2 , Δr_3

So, eq^n 1 becomes,



$$\vec{F} \cdot \Delta \vec{r} = \frac{m}{2} \frac{d}{dt}(v^2) \Delta t$$

- Now, we can integrate $(\Delta r_i \to 0)$

$$\int_{\vec{r}_a}^{\vec{r}_b} \vec{F}(\vec{r}) \cdot dr = \int_{t_a}^{t_b} \frac{m}{2} \frac{d}{dt}(v^2)\, dt$$

$$= \frac{1}{2} m v^2 \Big|_{t_a}^{t_b}$$

$$= \frac{1}{2} m v_b^2 - \frac{1}{2} m v_a^2$$

Here $v^2 = v_x^2 + v_y^2 + v_z^2$

So we have,

$$\underbrace{\int_{\vec{r}_a}^{\vec{r}_b} \vec{F} \cdot \Delta \vec{r}}_{\text{a line integral}} = \frac{1}{2} m v_b^2 - \frac{1}{2} m v_a^2$$

a line integral (Some examples later)

• Work-energy theorem: (multiple dimensions)

Our previous eq^n -

$$\underbrace{\int_{r_a}^{r_b} \vec{F} \cdot \Delta\vec{r}}_{\text{work done}} = \underbrace{\frac{1}{2} m v_b^2 - \frac{1}{2} m v_a^2}_{\text{difference in KE}}$$

So, work done by the force $= K_b - K_a$

$$W_{ba} = K_b - K_a$$

• General statement: The work ΔW done by the force $\vec{F}$ in a small displacement $\Delta\vec{r}$ is

$$\Delta W = \vec{F} \cdot \Delta\vec{r} = F \cos\theta \, \Delta r = F_{\parallel} \Delta r.$$
$$= K_b - K_a$$

where $F_{\parallel}$ is the parallel component of the force. $F_{\perp}$ does no work.

Q. What happens when several forces are involved?

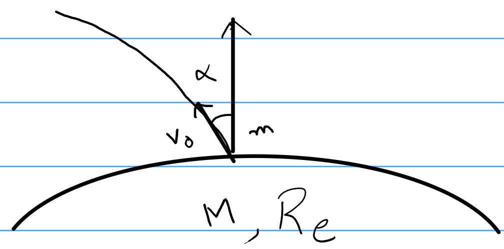
→ Homework problems. //

- Example: Escape velocity - the general case.

  - Now the mass is projected at an angle α from the vertical.

  - The force neglecting air resistence

$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$



$$= -mg\frac{R_e^2}{r^2}\hat{r}$$

where $g = GM/R_e^2$

→ a small incremental line element $d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$

$$\therefore \vec{F}\cdot d\vec{r} = -mg\frac{R_e^2}{r^2}\hat{r}\cdot(dr\,\hat{r} + r\,d\theta\,\hat{\theta})$$

$$= -mg\frac{R_e^2}{r^2}dr$$

$\hat{r}, \hat{\theta}$ → unit vectors.

$\left.\begin{array}{l}\hat{r}\cdot\hat{r} = 1 \\ \hat{r}\cdot\hat{\theta} = 0\end{array}\right\}$

So, from work-energy theorem,

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = -mgR_e^2\int_{R_e}^{r}\frac{dr}{r^2}$$

$$\Rightarrow \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = -mgR_e^2\left(\frac{1}{r} - \frac{1}{R_e}\right)$$

- The escape velocity is the value of v_0, for which $r = \infty$, $v = 0$

- So from the eq^n, $v_0 = \sqrt{2 g R_e}$ //

- **Conservative and nonconservative forces:**

Work energy th^m: $W_{ba} = K_b - K_a$

While we did the example

$$W_{ba} = \oint_a^b \vec{F} \cdot d\vec{r}$$

- Some cases require info. about only the initial and final places.

- Conservative forces → depends only on the initial and final position.

- Example: Work done by a uniform force:

$$W_{ba} = \int_{r_a}^{r_b} \vec{F} \cdot d\vec{r} \qquad \vec{F} = F_0 \hat{n}$$

$$W_{ba} = \int_{r_a}^{r_b} F_0 \hat{n} \cdot d\vec{r}$$

$$= F_0 \hat{n} \cdot \int_{r_a}^{r_b} d\vec{r}$$

$$= F_0 \hat{n} \cdot \left(\hat{i} \int_{x_a, y_a, z_a}^{x_b, y_b, z_b} dx + \hat{j} \int dy + \hat{k} \int dz \right)$$

$$= F_0 \hat{n} \cdot \left(\hat{i}\, x \Big|_{x_a}^{x_b} + \hat{j}\, y \Big|_{y_a}^{y_b} + \hat{k}\, z \Big|_{z_a}^{z_b} \right)$$

$$= F_0 \hat{n} \cdot \left(\hat{i}(x_b - x_a) + \hat{j}(y_b - y_a) + \hat{k}(z_b - z_a) \right)$$

$$= F_0 \hat{n} \cdot \left((\hat{i} x_b + \hat{j} y_b + \hat{k} z_b) - (\hat{i} x_a + \hat{j} y_a + \hat{k} z_a) \right)$$

$$= F_0 \hat{n} \cdot (\vec{r}_b - \vec{r}_a)$$

$$= F_0 |\vec{r}_b - \vec{r}_a| \cos\theta$$

$$= F_0 \cos\theta\, |\vec{r}_b - \vec{r}_a|$$

$$W_{ba} = F_{\parallel} |\vec{r}_b - \vec{r}_a| \quad //$$

So, the work done by a constant force depends only on the endpoints, and not on the path.

Ex: Central force

- **Potential energy:**

Work done by a conservative force depends only on the end points. For a conservative fore,

$$\int_{r_a}^{r_b} \vec{F} \cdot d\vec{r} = f(r_b) - f(r_a)$$

$$= -U(r_b) + U(r_a)$$

$U \rightarrow$ Potential energy

Now from the work energy theorem,

$$W_{ba} = -U_b + U_a = K_b - K_a$$

Rearrange, $K_a + U_a = K_b + U_b$

$$K_a + U_a = K_b + U_b = E$$

$E \rightarrow$ Energy

- If the force is conservative, then the total energy of the particle is independent of the position of the particle.

$\rightarrow$ it remains constant.

Law of conservation of energy.

$$U = U + b$$

$$F = -\frac{dU}{dx}$$

$$F = -\frac{dU}{dx}$$