Lecture-2 : Work and energy

- Fundamental problem of mechanics : predicting the motion of a system under known interactions.

- In order to do that → we integrate $\vec{F} = m\vec{a}$.
($\vec{a} \to \int \to \vec{v} \to \int \to \vec{x}$). This needs $F = F(t)$ and $\vec{F}$ is not usually known as a function of 't'. It is usually $F = F(x)$, function of position. eg ($F = -kx$).

- But we are generally interested in knowing how the force varies with position.

- So we need to integrate →

$$m \frac{d\vec{v}}{dt} = \underbrace{\vec{F}(\vec{r})}_{\text{Function of position.}} \qquad (1)$$

- Now in order to do so, we would learn some awesome concepts. Such as → work energy theorem and law of conservation of energy.

Note: In general eq. (1) can be integrated — i.e. the fundamental problem of mechanics reduces to one that of solving a 2nd order ordinary diff. eqn. However, such an approach is too narrow and affords too little physical understanding.

- Integrating EoM in one dimension :

Basic eq^n $\rightarrow$ $m \dfrac{d^2x}{dt^2} = F(x)$

$$\Rightarrow m \frac{dv}{dt} = F(x)$$

We can solve it for v using a simple trick. But first, let's integrate

$$\underbrace{m \int_{x_a}^{x_b} \frac{dv}{dt}\, dx}_{\text{Change variable } x \text{ to } t.} = \underbrace{\int_{x_a}^{x_b} F(x)\, dx}_{\text{can be integrated by standard methods, since } F(x) \text{ is known.}}$$

Change variable x to t.

$\hookrightarrow$ can be integrated by standard methods, since $F(x)$ is known.

So,

$$dx = \left(\frac{dx}{dt}\right) dt$$

$$\Rightarrow dx = v\, dt$$

So again using this,

$$m \int_{x_a}^{x_b} \frac{dv}{dt}\, dx = m \int_{t_a}^{t_b} \frac{dv}{dt}\, v\, dt$$

$$= m \int_{t_a}^{t_b} \frac{d}{dt}\left(\frac{1}{2} v^2\right) dt$$

$$= \frac{1}{2} m v^2 \Big|_{t_a}^{t_b}$$

$$= \frac{1}{2} m v_b^2 - \frac{1}{2} m v_a^2$$

where, $x_a \equiv x(t_a)$, $v_a = v(t_a)$ ...

Now putting everything together,

$$\frac{1}{2} m v_b^2 - \frac{1}{2} m v_a^2 = \int_{x_a}^{x_b} F(x)\, dx$$

Alternatively, using indefinit upper limits,

$$\frac{1}{2} m v^2 - \frac{1}{2} m v_a^2 = \int_{x_a}^{x} F(x)\, dx \quad \text{— (11)}$$

where $v = v(x)$

- Eq^n (11) gives us 'v' as a function of x. Since $v = \frac{dx}{dt}$, we can solve (11) for $\frac{dx}{dt}$ and integrate again to find $x(t)$.

Example: Mass thrown upward in a uniform gravitational field

mass $\rightarrow m$ $\qquad \uparrow z$, v_0 $\qquad$ gravitational force = constant, no air friction

$$F = -mg$$

$$\frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2 = \int_{z_0}^{z_1} F\,dz$$

$$= \int_{z_0}^{z_1} -mg\,dz$$

$$= -mg\int_{z_0}^{z_1} dz$$

$$= -mg\,z\Big|_{z_0}^{z_1}$$

$$\frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2 = -mg(z_1 - z_0)$$

$v_1 = 0$ at the maximum height.

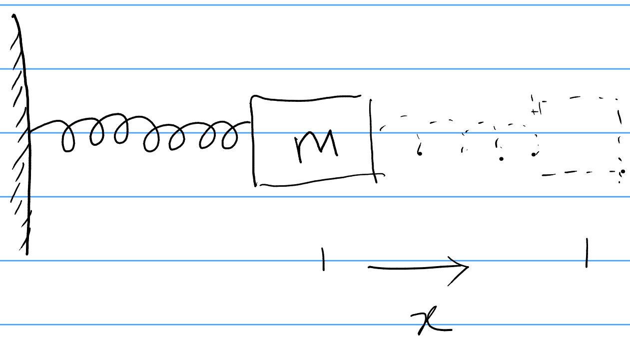
$$+\frac{1}{2}\cancel{m}v_0^2 = +\cancel{m}g(z_1 - z_0)$$

$$\frac{1}{2}v_0^2 = g(z_1 - z_0)$$

$$\frac{v_0^2}{2g} = z_1 - z_0$$

$$\Rightarrow \boxed{z_1 = z_0 + \frac{v_0^2}{2g}}$$

- EOM of simple harmonic motion.



$$F = -kx$$

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = -k\int_{x_0}^{x} x\,dx$$

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = -\frac{1}{2}kx^2 + \frac{1}{2}kx_0^2$$

Initial cond^{n}s : $t=0$, $v_0 = 0$, x_0

$$v^2 = -\frac{k}{m}x^2 + \frac{k}{m}x_0^2$$

Now,

$$v = \frac{dx}{dt}$$

$$v = \sqrt{\frac{k}{M}}\sqrt{x_0^2 - x^2}$$

$$\Rightarrow \frac{dx}{dt} = \sqrt{\frac{k}{M}}\sqrt{x_0^2 - x^2}$$

$$\Rightarrow \int_{x_0}^{x} \frac{dx}{\sqrt{x_0^2 - x^2}} = \int_0^t \sqrt{\frac{k}{M}}\, dt$$

$$\Rightarrow \int_{x_0}^{x} \frac{dx}{\sqrt{x_0^2 - x^2}} = \sqrt{\frac{k}{M}}\, t$$

$$\Rightarrow \arcsin\left(\frac{x}{x_0}\right)\Bigg|_{x_0}^{x} = \omega t \qquad \omega = \sqrt{\frac{k}{m}}$$

$$\Rightarrow \arcsin\left(\frac{x}{x_0}\right) - \arcsin(1) = \omega t$$

$$\Rightarrow \arcsin\left(\frac{x}{x_0}\right) - \frac{\pi}{2} = \omega t$$

$$\Rightarrow \arcsin\left(\frac{x}{x_0}\right) = \omega t + \frac{\pi}{2}$$

$$\Rightarrow \frac{x}{x_0} = \sin\left(\omega t + \pi/2\right)$$

$$\Rightarrow x = x_0 \sin\left(\omega t + \pi/2\right)$$

$$\Rightarrow \boxed{x = x_0 \cos(\omega t)}$$

Eq^n for SHM.

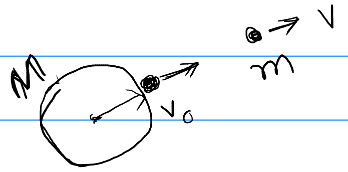
- Work-energy theorem: (one dimension)

$$\underbrace{\frac{1}{2}Mv_b^2 - \frac{1}{2}Mv_a^2}_{\text{Kinetic energy }(K)} = \underbrace{\int_{x_a}^{x_b} F(x)\,dx}_{\text{Work }(W)}$$

$$\boxed{W_{ba} = K_b - K_a}$$

$\boxed{\text{unit of work - Joule (J)}}$

• Example : Inverse square field $F \propto \frac{1}{r^2}$



$$F = -\frac{GMm}{r^2}$$

$R_e \Rightarrow$ radius of earth

$$K(r) - k(r_0) = \int_{R_e}^{r} F(r)\, dr$$

$$= -GMm \int_{R_e}^{r} \frac{1}{r^2}\, dr$$

$$\frac{1}{2} m v(r)^2 - \frac{1}{2} m v_0^2 = -GMm\left(\frac{1}{r} - \frac{1}{R_e}\right)$$

Max height $v(r) = 0$

$$v_0^2 = 2GM\left(\frac{1}{R_e} - \frac{1}{r}\right) \qquad g = \frac{GM}{R_e^2}$$

$$= 2gR_e^2\left(\frac{1}{R_e} - \frac{1}{r}\right)$$

$$v_0^2 = 2gR_e\left(1 - \frac{R_e}{r}\right)$$

$$v_0^2 = 2gR_e\left(1 - \frac{R_e}{r_{max}}\right)$$

$$\Rightarrow r_{max} = \frac{R_e}{1 - \frac{v_0^2}{2gR_e}}$$

$$1 = \frac{v_0^2}{2gR_e}$$

$$\Rightarrow v_0 = \sqrt{2gR_e}$$

$$= \sqrt{2 \times 9.8 \times 6.4 \times 10^6}$$

$$= 11.2 \times \sqrt{10^6} = 11.2 \times 10^3 \ m/s$$

$$\boxed{v_0 = 11.2 \ km/s}$$

escape velocity

- Calculate the energy needed to eject a 1500 kg spacecraft from the surface of earth.