Lecture 10: Central force and planetary motion

$m_1 \quad m_2$

$\rightarrow f(r) \leftarrow$

Energy eq^n and energy diagram

- We found two equivalent ways of writing E in the center of mass system.

$$E = \frac{1}{2}\mu v^2 + U(r) \qquad \text{——— (1)}$$

and,

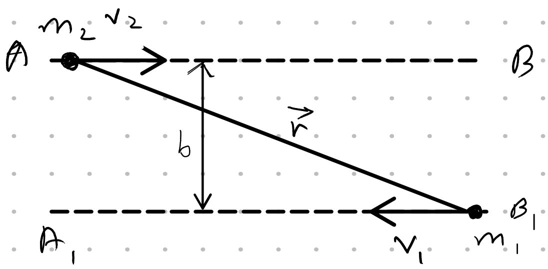
$$E = \frac{1}{2}\mu \dot{r}^2 + U_{eff}(r) \qquad \text{——— (2)}$$

- Eq^n (1) is handy for evaluating E.
- Eq^n (2) depends on single coordinate r. Good to solve. Because θ is suppressed. As

$$\frac{1}{2}\mu(r\dot{\theta})^2 = \frac{l^2}{2\mu r^2}$$

and $$U_{eff}(r) = \frac{l^2}{2\mu r^2} + U(r)$$

- **Example:** Two non-interacting particles m_1 and m_2 move towards each other with velocity $\vec{v}_1$ and $\vec{v}_2$. Their paths are offset by a distance b.



- Non-interacting, so $f(r) = 0$
- Relative velocity,

$$\vec{v}_0 = \dot{\vec{r}} = \dot{\vec{r}}_1 - \dot{\vec{r}}_2$$
$$= \vec{v}_1 - \vec{v}_2 = \text{constant.}$$

- Total energy,

$$(1) \Rightarrow E = \frac{1}{2}\mu v_0^2 + U(r) = \frac{1}{2}\mu v_0^2$$

- Effective potential,

$$U_{eff} = \frac{l^2}{2\mu r^2} + U(r)$$

$$U_{eff} = \frac{l^2}{2\mu r^2} + U(r) = \frac{l^2}{2\mu r^2}$$

- Hence $(2) \Rightarrow E = \frac{1}{2}\mu \dot{r}^2 + \frac{l^2}{2\mu r^2} = \frac{1}{2}\mu v_0^2$

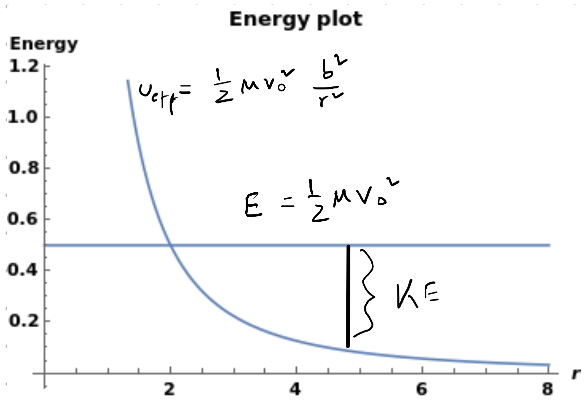
$(1) = (2)$

- When m_1 and m_2 pass each other, $r = b$ and $\dot{r} = 0$.

So, $$E = \frac{l^2}{2\mu b^2} = \frac{1}{2}\mu v_0^2$$

$$\Rightarrow \boxed{l = \mu b v_0}$$

and, $$U_{eff}(r) = \frac{1}{2}\mu v_0^2 \frac{b^2}{r^2}$$



- Kinetic energy associated with radial motion →

$$K = \frac{1}{2}\mu \dot{r}^2 = E - U_{eff}$$

$U_q \propto \frac{1}{r^2}$

$r = b$

- KE is never negative. So motion is restricted to the region where, $E - U_{eff} \geq 0$.

- Initially r is very large. As the particles approach, the KE decreases vanishing at turning point $(\dot{r} = 0)$, r_t. Here the motion is purely tangential.

- At the turning point, $E = U_{eff}(r_t)$

$$\frac{1}{2}\mu v_0^2 = \frac{1}{2}\mu v_0^2 \frac{b^2}{r_t^2} \Rightarrow r_t = b.$$

• Planetary motion :

- Motion of earth around the sun. Or motion of Satellites around earth.

- In this case, $f(r) \neq 0$.

$$f(r) = -\frac{G m_1 m_2}{r^2}$$

$$U_{eff} = \frac{l^2}{2\mu r^2} + U(r)$$

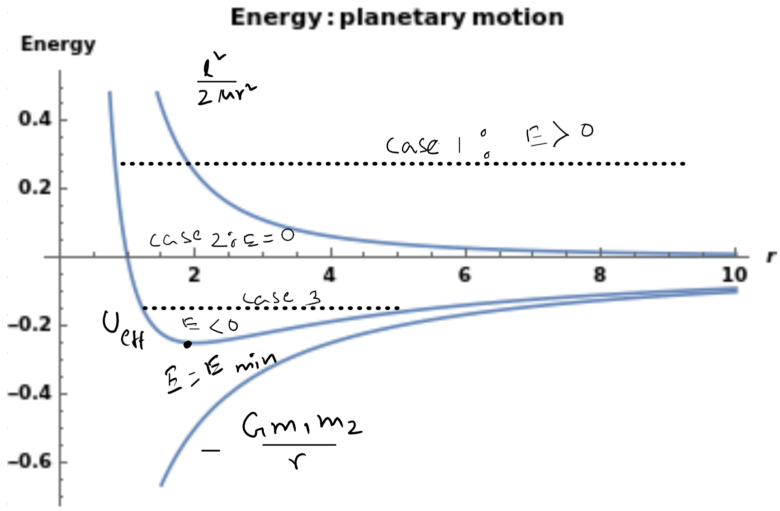
$$U(r) = -\frac{G m_1 m_2}{r}$$

- So the effective potential, $U_{eff} = -\frac{G m_1 m_2}{r} + \frac{l^2}{2\mu r^2}$

- If $l \neq 0$, repulsive centrifugal potential $l^2/2\mu r^2$ dominates at small r.

- The plot shows various values of total energy,

- $KE = E - U_{eff}$, motion is restricted to the region where, $KE \geq 0$.



- The nature of motion is determined by total energy E.

$\longrightarrow$
- Case 1 : $E > 0$
  - r is unbounded for large values. Must exceed a certain minimum if $l \neq 0$. particles are kept apart by "centrifugal barrier".
  - Hyperbolic orbits.

- Case 2 : $E = 0$
  - Similar to case 1. Boundary between bounded and unbounded motion.
  - Parabolic orbit.

- Case 3 : $E < 0$
  - The motion is bounded for both small and large r. Bound system.
  - Elliptical orbit.

- Case 4 : $E = E_{min}$
  - r is restricted to 1 value. Particles stay at a constant distance.
  - circular orbit.

• Solving planetary motion :

$M = S\,m$
$m = E\,m$

$c = GMm$

- We have the potential

$$U(r) = -G\frac{Mm}{r} = -\frac{c}{r}.$$

$M \rightarrow$ Mass of sun/planet

$m \rightarrow$ Mass of planet/satellite

- We have from previous lecture,

$\dot{r} = (\ )$
$\dot{\theta} = (\ )$
$\frac{d\theta}{dr} = (\ )$

$$\frac{d\theta}{dr} = \frac{l}{\mu r^2} \times \frac{1}{\left\{(2/\mu)(E - U_{eff})\right\}^{1/2}}$$

$$\Rightarrow \theta - \theta_0 = \int_{\theta_0}^{\theta} d\theta = l\int \frac{dr}{r(2\mu E r^2 + 2\mu c r - l^2)^{1/2}} \quad \text{——} (a)$$

here $\theta_0 \rightarrow$ constant of integration.

- The integral in (a) can obtained from integration table. The result is,

$$\theta - \theta_0 = \mathrm{ArcSin}\left[\frac{\mu c r - l^2}{r(\mu^2 c^2 + 2\mu E l^2)^{1/2}}\right]$$

$$\Rightarrow \mu c r - l^2 = r\left(\mu^2 c^2 + 2\mu E l^2\right)^{1/2} \sin(\theta - \theta_0)$$

Now solving for r,

$$\Rightarrow r = \frac{(l^2/\mu c)}{1 - \left(1 + (2El^2/\mu c^2)\right)^{1/2} \sin(\theta - \theta_0)}$$

Here, θ_0 is usually taken as $\pi/2$

- For further simplification, we define two parameters

$$r_0 \equiv \frac{l^2}{\mu c}$$ (r_0 is the radius of the circular orbit for correspondin l, μ, c)

$$\epsilon \equiv \sqrt{1 + \frac{2El^2}{\mu c^2}}$$ (ϵ is the eccentricity characterizes the shape of the orbit.)

- Use these two parameters in the previous eq^n, it becomes,

$$r = \frac{r_0}{1 - \epsilon \cos\theta}$$

x, y

- This eq^n looks more familiar in Cartesian coordinates. (x, y).

$$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$$
$$y = r\sin\theta$$

So, $$r = \frac{r_0}{1 - \epsilon\cos\theta}$$

$$\Rightarrow r - \epsilon r\cos\theta = r_0$$

$$\Rightarrow \sqrt{x^2+y^2} - \epsilon x = r_0$$

$$\Rightarrow x^2 + y^2 = (r_0 + \epsilon x)^2$$

$$\Rightarrow x^2 + y^2 = r_0^2 + \epsilon^2 x^2 + 2\epsilon r_0 x$$

$$\Rightarrow \boxed{(1-\epsilon^2)x^2 + y^2 - 2\epsilon r_0 x = r_0^2}$$

This is the eq^n of orbits for planetary motion. (General)

• Now different cases -

$$\boxed{(1-\epsilon^2)x^2 - 2r_0\epsilon x + y^2 = r_0^2}$$

• Case ①, $\epsilon > 1$: $(E > 0)$

- Coefficients of x^2 and y^2 are unequal and opposite in sign.
- So it will have this form, $y^2 - Ax^2 - Bx = \text{const.}$
- This is the eq^n of of a hyperbola.

• Case ②, $\epsilon = 1$: $(E = 0)$

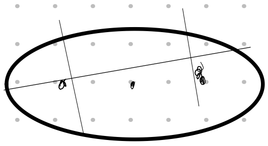
- Eq^n becomes,

$$x = \frac{y^2}{2r_0} - \frac{r_0}{2}$$

- This is the eq^n of a parabola.

• Case ③, $0 \leq \epsilon < 1$: $\left(-\frac{mc^2}{2l^2} \leq E < 0\right)$

- Coefficients of x^2 and y^2 are unequal, but has the same sign.
- Eq^n : $y^2 + Ax^2 - Bx = \text{const.}$



- Eq^n of an ellipse. (geometric center not at the origin of coord. system).

• Case ④, $\epsilon = 0$: $\left(E = -\frac{c\mu^2}{2l^2}\right)$

- Eq^n becomes, $x^2 + y^2 = r_0^2$
- Circular orbit.
- (Special case of ellipse)

• Elliptial orbits :

- Elliptical orbits are very important. ($E < 0$, $0 \leq \epsilon < 1$)

- The orbit eqn is,

1 + ϵ

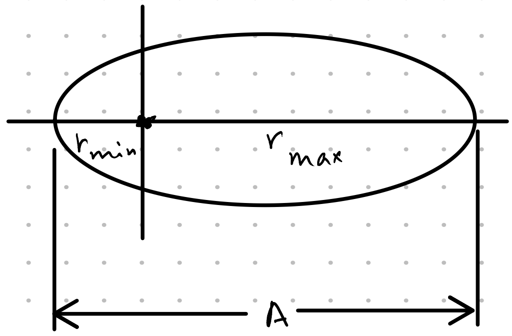
$$r = \frac{r_0}{1 - \epsilon \cos\theta}$$

- maximum value of r occurs at $\theta = 0$,

$$r_{max} = \frac{r_0}{1-\epsilon}$$

- minimal value of r occurs at $\theta = \pi$,

$$r_{min} = \frac{r_0}{1+\epsilon}$$



- length of the major axis,

$$A = r_{min} + r_{max}$$

$$= r_0\left(\frac{1}{1+\epsilon} + \frac{1}{1-\epsilon}\right)$$

$$\Rightarrow A = \frac{2r_0}{1-\epsilon^2}$$

Now, $r_0 = l^2/Mc$, $\epsilon = \sqrt{1 + \frac{2El^2}{Mc^2}}$

$$\Rightarrow A = \frac{2l^2/Mc^2}{1 - \left(1 + 2El^2/Mc^2\right)}$$

$$\Rightarrow \boxed{A = \frac{C}{-E}}$$

- This means shape of the major axis is independent of l. Orbits with the same major axis has the same energy.

- The ratio of r_{max} and r_{min},

$$\frac{r_{max}}{r_{min}} = \frac{r_0/(1-\epsilon)}{r_0/(1+\epsilon)} = \frac{1+\epsilon}{1-\epsilon}$$

- When, $\epsilon \to 0$, $r_{max}/r_{min} \simeq 1$. So the ellipse becomes a circle.

- When $\epsilon \to 1$, the ellipse become elongated.

- All the planets in solar system are in elliptical orbit.

• Keplar's laws :

- Johannes keplar was an assistant of Tycho Brahe.
- Tycho Brahe recorded data for orbits of planets using telescope.
- Keplar took this data and formulated the laws.
- The laws are :

  ① Each planet moves in an ellipse with the sun at its focus.

  ② The radius vector from the sun to a planet sweeps out equal area in equal times.

  ③ The period of revolution T of a planet about the sun is related to the major axis of ellipse A by

  $$T^2 = K A^3$$

  Where k is same for all planets.

- We already saw 1st law in the previous section.

- **Proof of 2nd law:**

In planetary motion, $U(r) = -\frac{c}{r}$

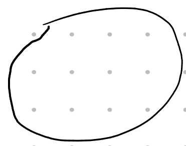
and we have,

$$E = \frac{1}{2}\mu\dot{r}^2 + U_{eff}$$

$$\Rightarrow \frac{dr}{dt} = \sqrt{\frac{2}{\mu}\left(E - U_{eff}\right)}$$

$$\Rightarrow t_b - t_a = \int_{r_b}^{r_a} \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - U_{eff}\right)}}$$

$$\Rightarrow t_b - t_a = \mu \int_{r_a}^{r_b} \frac{r\,dr}{\left(2\mu E r^2 + 2\mu C r - l^2\right)^{1/2}}$$



It is a standard integral — Sol^n is,

$$t_b - t_a = \left.\frac{\sqrt{2\mu E r^2 + 2\mu C r - l^2}}{2E}\right|_{r_a}^{r_b} - \left(\frac{\mu C}{2E}\right)\frac{1}{\sqrt{-2\mu E}} \left.\arcsin\left(\frac{-2\mu E r - \mu c}{\sqrt{\mu^2 c^2 + 2\mu E l^2}}\right)\right|_{r_a}^{r_b}$$

For a complete period, $\underbrace{t_b - t_a = T \quad \text{and} \quad r_b = r_a}$

- The first term in rhs vanishes.
- The second term - arcsin changes by 2π

So the result,

$$T = \frac{\pi \mu C}{-E} \frac{1}{\sqrt{-2\mu E}}$$

$$\Rightarrow T^2 = \frac{\pi^2 \mu\, c^2}{(-2E^3)} \qquad \left(\text{using } A = -\frac{c}{E}\right)$$

$$\Rightarrow T^2 = \left(\frac{\pi^2 \mu}{2c}\right) A^3 \qquad \Rightarrow \boxed{T^2 = k A^3}$$